Cours du 10.05.2021

lundi, 10 mai 2021 18:39

Fonctions de probabilité

$$\underbrace{\mathbb{P} : \Omega \longmapsto [0,1]}_{\text{Loi de probabilité}}$$

Discret : $\sum_{x\in\Omega} \mathbb{P}(x) = 1$

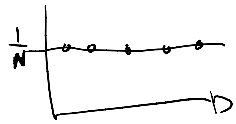
Continu : $\int_{x\in\Omega} \mathbb{P}(x)\,dx = 1$

Aire totale $= 1$!

Loi uniforme

$\Omega = \{1, 2, \dots, N\}$ $\quad \mathbb{P}(x) = \frac{1}{N}$

max - min

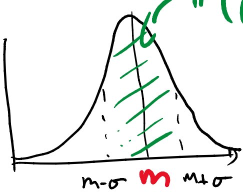


Loi Normale (courbe gaussienne)

moyenne

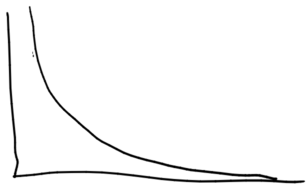
$$\mathbb{P}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

écart-type

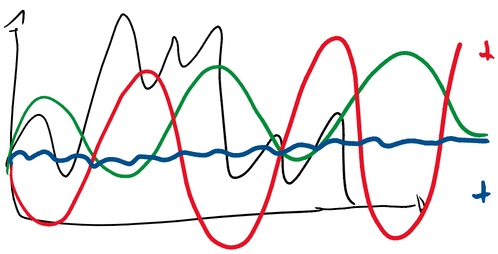


$\mathbb{P}([m-\sigma, m+\sigma]) \simeq 68\%$

Loi exponentielle



Loi binomiale, loi Multinomiale, Loi de Poisson, Loi de Bernouilli.



Approximer les fonctions de probabilité complexes avec des "combinaisons" des lois connues.

Exemple de lancé de dé équilibré

$\Omega = \{1, 2, 3, \dots, 6\}$

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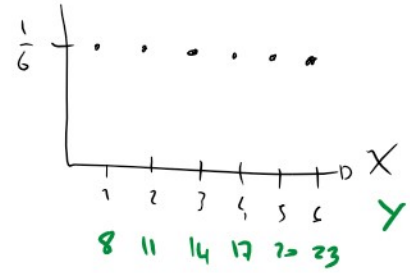
$$\Omega = \{1, 2, 3, \dots, 6\}$$

$$\mathbb{P}(x) = \frac{1}{6} \quad x \in \Omega$$

Au lieu d'observer x (la valeur du dé), j'observe $\cancel{y=}3x+5$

$$\begin{array}{c} x = \; 1 \;\; 2 \;\; \boxed{3} \;\; 4 \;\; 5 \;\; 6 \\ \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \\ \Omega' = \{8, 11, \boxed{14}, 17, 20, 23\} \end{array}$$



$$\mathbb{P}'(y=14) = \mathbb{P}(x=3) = \frac{1}{6}$$

$$\mathbb{P}(\alpha \cdot x + c) = \mathbb{P}(x)$$

Exemple 2 : La somme de 2 Lancés de dés équilibrés

Loi 1

Dé 1

$$\Omega_1 = \{1, 2, \dots, 6\}$$

$$\mathbb{P}_1(x_1) = \frac{1}{6} \quad x_1 \in \Omega_1$$

Loi 2

Dé 2

$$\Omega_2 = \{1, 2, \dots, 6\}$$

$$\mathbb{P}_2(x_2) = \frac{1}{6} \quad x_2 \in \Omega$$

Loi $Y = X_1 + X_2$

Somme

$$\Omega_Y = \{2, 3, 4, \dots, 12\}$$

$$\mathbb{P}_Y(y) = ?$$

D1/D2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

36 combinaisons possibles

(6×6)

$$\mathbb{P}_Y(2) = \frac{1}{36} = \mathbb{P}_Y(1 \text{ et } 1) = \underbrace{\mathbb{P}_1(x_1=1)}_{1/6} \cdot \underbrace{\mathbb{P}_2(x_2=2)}_{1/6}$$

$$\mathbb{P}_Y(12) = \frac{1}{36} \quad \dots\dots$$

$$\mathbb{P}(3) = \mathbb{P}_Y(1+2) + \mathbb{P}(2+1) = \frac{1}{\,} \cdot 1 \quad , \quad 1 \quad 1 \qquad 2$$

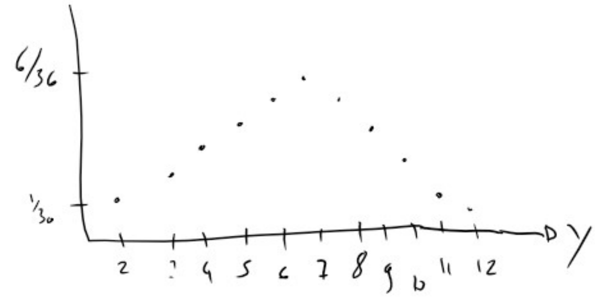
$$\mathbb{P}_y(3) = \mathbb{P}_y(1+2) + \mathbb{P}_y(2+1) = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{2}{36} = \mathbb{P}_y(11)$$

$$\mathbb{P}_y(4) = \mathbb{P}_y(10) = \frac{3}{36}$$

$$\mathbb{P}_y(5) = \mathbb{P}_y(9) = \frac{4}{36}$$

$$\mathbb{P}_y(6) = \mathbb{P}_y(8) = \frac{5}{36}$$

$$\mathbb{P}_y(7) = \frac{6}{36}$$



Avec 3 dés: $\mathbb{P}(3) = \mathbb{P}(18) = \frac{1}{216} = \frac{1}{6^3}$

$$\mathbb{P}(4) = \mathbb{P}(112) + \mathbb{P}(121) + \mathbb{P}(211) = \frac{3}{216}$$

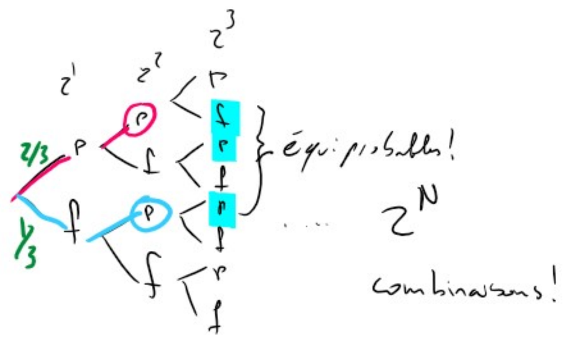
$$\mathbb{P}(5) = \frac{6}{216}$$

$1+1+3 \Rightarrow 3$ perm [113, 131, 311]

$1+2+2 \Rightarrow 3$ perm

On touche ici les notions du calcul COMBINATOIRE !
#permutations de N éléments = N! (l'ordre compte!!!)

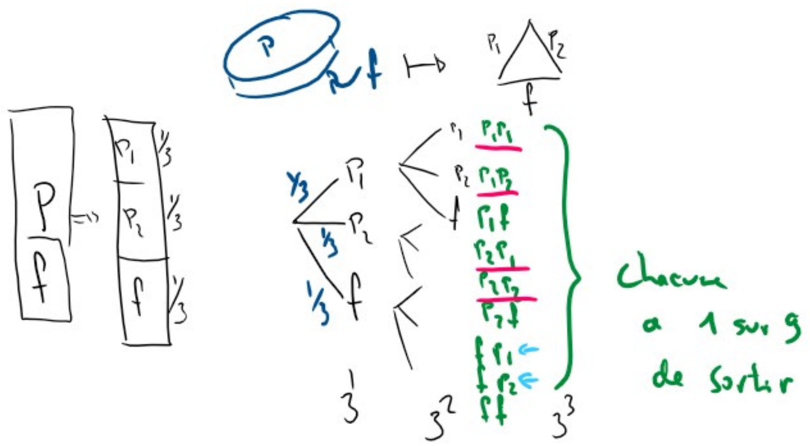
Pièce "déséquilibré" qu'on tire N fois avec



$\mathbb{P}(\text{pile}) = \frac{2}{3}$ $\qquad \mathbb{P}(\text{face}) = \frac{1}{3}$

$\longmapsto$

Prob. se MULTIPLIENT!



(9) (27)

Pile = p_1 ou p_2

$$\mathbb{P}(pp) = \mathbb{P}(p_1p_1) + \mathbb{P}(p_1p_2) + \mathbb{P}(p_2p_1) + \mathbb{P}(p_2p_2)$$

$$= \frac{4}{9} = \mathbb{P}(p) \times \mathbb{P}(p) = \frac{2}{3} \times \frac{2}{3}$$

$$\mathbb{P}(fp) = \mathbb{P}(fp_1) + \mathbb{P}(fp_2) = \frac{2}{9}$$

$$= \mathbb{P}(f) \cdot \mathbb{P}(p) = \frac{1}{3} \cdot \frac{2}{3}$$

$$\mathbb{P}(pf) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

$$\mathbb{P}(ff) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

3 Tirages de la pièce déséquilibrée

$$\mathbb{P}(ppp) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

$$\mathbb{P}(ppf) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$$

$$\mathbb{P}(pff) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27}$$

$$\mathbb{P}(fff) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

Quelle est la probabilité de faire k fois pile sur N=3 lancés ?

$p = 1$ $\quad$ X_1 lancé 1

$f = 0$ $\quad$ X_2 lancé 2

X_3 lancé 3

$$Y = X_1 + X_2 + X_3$$

$$\Omega_y = \{3, 2, 1, 0\}$$

$$\mathbb{P}_y(3) = 1 \cdot \mathbb{P}(ppp) = 1\left(\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\right) = \frac{8}{27} = \frac{8}{27}$$

$\left.\begin{matrix} ppf \\ pfp \\ fpp \end{matrix}\right\}$ $$\mathbb{P}_y(2) = 3\cdot\mathbb{P}(ppf) = 3\left(\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\right) = \frac{3\cdot 4}{27} = \frac{12}{27}$$

$$\mathbb{P}_y(1) = 3\cdot\mathbb{P}(ffp) = 3\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{3\cdot 2}{27} = \frac{6}{27}$$

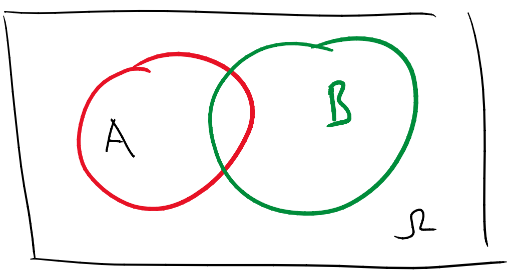
$$1\cdot\mathbb{P}_y(0) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27} = \frac{1}{27}$$

$\frac{15}{27}$ $\quad \frac{27}{27} = 1$!

⚠ $\mathbb{P} \neq 1$

FAUX !

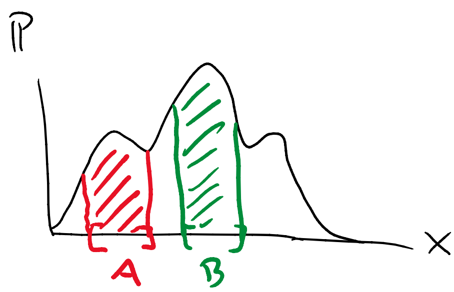
Calculs avec les probabilités et les sous-ensembles !



$$\mathbb{P}(\Omega) = 1$$

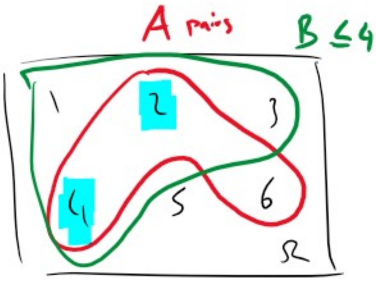
$$\mathbb{P}(A) = \sum_{x \in A} \mathbb{P}(x) \qquad \text{discret}$$

$$\mathbb{P}(A) = \int_{x \in A} \mathbb{P}(x)\,dx \qquad \text{Continu}$$



$$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}$$

$A \cap B$



$$A = \{ x \mid x \text{ est pair} \}$$

$$B = \{ x \mid x \leq 4 \}$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$= \underbrace{3 \cdot \frac{1}{6}}_{\mathbb{P}(A)} + \underbrace{4 \cdot \frac{1}{6}}_{\mathbb{P}(B)} - 2 \cdot \frac{1}{6} = \frac{5}{6}$$

Si $A \subseteq \Omega$ $\boxed{\mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A)}$